

CONTINUOUS TIME PORTFOLIO CHOICE UNDER MONOTONE PREFERENCES WITH QUADRATIC PENALTY - STOCHASTIC INTEREST RATE CASE

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ABSTRACT. This is a follow up of our previous paper - Trybuła and Zawisza [17], where we considered a modification of a monotone mean-variance functional in continuous time in stochastic factor model. In this article we address the problem of optimizing the mentioned functional in a market with a stochastic interest rate. We formulate it as a stochastic differential game problem and use Hamilton-Jacobi-Bellman-Isaacs equations to derive the optimal investment strategy and the value function.

1. INTRODUCTION

Mean-variance analysis, introduced by Markowitz [11], has long been a popular approach to determine the structure and composition of an optimal portfolio. Nevertheless, it is well known that mean-variance functional is not monotone and this is a serious drawback. Namely, Maccheroni et al. [10] gave a simple example when an investor with mean-variance preferences may strictly prefer less to more, thus violating one of the most compelling principles of economic rationality. For this reason, they created a new class of monotone preferences that coincide with mean-variance preferences on their domain of monotonicity, but differ where mean-variance preferences fail to be monotone and are therefore not economically meaningful.

A modification of Maccheroni type objective function has been first studied in detail in a dynamic optimization framework by Trybuła and Zawisza [17]. They showed that the solution to the problem of optimizing the mentioned functional in stochastic factor model coincides with the solution to classical Markowitz problem with a suitably chosen risk aversion coefficient.

In this work we assume that an investor has access to the market, where he can freely invest in a bank account and a risky asset. Moreover, we suppose that the interest rate is given by a solution to a stochastic differential equation driven by one dimensional Brownian motion. The purpose is to describe an optimal financial strategy which an investor can follow in order to maximize his performance criterion which is given by the modification of the monotone mean-variance functional from Trybuła and Zawisza [17].

The paper is structured in the same manner as mentioned article but the problem here is much harder to solve and results cannot be obtained using direct reasoning from that work.

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For literature review about finite horizon max-min problems we refer to Bordigoni et al. [3], Hernández and Schied [8], Mataramvura and Øksendal [12], Øksendal and Sulem [15], Trybuła and Zawisza [17] and Zawisza [19].

For interesting discussion about optimal investment in the presence of stochastic interest rate see for instance Bielecki and Pliska [1] and [2], Brennan and Xia [4], Korn and Kraft [9], Munk and Sørensen [13] and [14]. It is worth mentioning here the paper of Flor and Larsen [6], where robust utility maximization with homothetic penalty function is considered.

2. GENERAL MODEL DESCRIPTION

Let (Ω, \mathcal{F}, P) be a probability space with a filtration $(\mathcal{F}_t, 0 \leq t \leq T)$ possibly enlarged to satisfy usual assumptions and generated by Brownian motion $(W_t, 0 \leq t \leq T)$ defined on (Ω, \mathcal{F}, P) . Suppose that an investor has access to the market with a bank account $(B_t, s \leq t \leq T)$ and a risky asset $(S_t, s \leq t \leq T)$. Moreover, we assume that the interest rate is given by a stochastic process $(r_t, s \leq t \leq T)$ and particularly we will consider the case of the Vasicek model for short rate. Processes mentioned above are solutions of the following system of stochastic differential equations

$$(2.1) \quad \begin{cases} dB_t = r_t B_t dt, \\ dS_t = (r_t + \lambda(r_t)\sigma(r_t, t))S_t dt + \sigma(r_t, t)S_t dW_t, \\ dr_t = \bar{\mu}(r_t)dt + \bar{\sigma}(r_t)dW_t, \quad r_s = r > 0, \end{cases}$$

where the coefficients $\sigma > 0$, λ , $\bar{\mu}$, $\bar{\sigma}$ are continuous functions and they are assumed to satisfy all the required regularity conditions, in order to guarantee that the unique strong solution to (2.1) exists. For notational convenience and in order to ensure that under general conditions there exists a solution to equation (2.17) we avoid putting "t"-dependence in model coefficients. The exception is in the case of function σ , since the risky asset S_t has a natural interpretation as a price of bond in a short term one factor interest rate models.

It is still possible to generalize this model by adding additional Brownian motion to the dynamics of S_t , however we are not able to solve the resulting HJBI equation, which is much extended even in current problem.

The rest of the setting is exactly the same as in Trybuła and Zawisza [17]. Namely, we will consider the class of P -equivalent measures

$$\mathcal{Q} := \left\{ Q \sim P : \frac{dQ}{dP} = \mathcal{E} \left(\int \eta_t dW_t \right)_T, \quad \eta \in \mathcal{M} \right\},$$

where $\mathcal{E}(\cdot)_t$ denotes the Doleans-Dade exponential and \mathcal{M} is the set of all progressively measurable processes η taking values in \mathbb{R} , such that

$$\mathbb{E} \left(\frac{dQ^\eta}{dP} \right)^2 < +\infty \quad \text{and} \quad \mathbb{E} \left(\frac{dQ^\eta}{dP} \right) = 1,$$

where Q^η denotes the measure determined by $\eta \in \mathcal{M}$. Moreover, let us define additional family of stochastic processes $(Y_t^\eta, s \leq t \leq T)$ which are given by the stochastic differential

equations

$$dY_t^\eta = \eta_t Y_t^\eta dW_t, \quad Y_s^\eta = y > 0, \quad \eta \in \mathcal{M}.$$

Then, notice that

$$Y_T^\eta = y \frac{dQ^\eta}{dP}, \quad \eta \in \mathcal{M}.$$

Let $(X_t^\pi, s \leq t \leq T)$ be the investors wealth process with the following dynamics

$$(2.2) \quad dX_t^\pi = (\pi_t \lambda(r_t) \sigma(r_t, t) + r_t X_t^\pi) dt + \pi_t \sigma(r_t, t) dW_t, \quad X_s^\pi = x > 0,$$

where x denotes a current wealth of the investor, whereas a control π_t we can interpret as a part of wealth invested in S_t . Note that π_t as well as the portfolio wealth X_T^π are allowed to be negative.

Definition 2.1. A control (or strategy) $\pi = (\pi_s, t \leq s \leq T)$ is admissible on the time interval $[t, T]$, written $\pi \in \mathcal{A}_{x,y,r,t}$, if it satisfies the following assumptions:

- (i) π is progressively measurable;
- (ii) unique solution to (2.2) exists and

$$\mathbb{E}_{x,y,r,t}^\eta \left[\sup_{t \leq s \leq T} |X_s^\pi| \right] < +\infty \quad \text{for all } \eta \in \mathcal{M},$$

where \mathbb{E}^η denotes the expectation with respect to measure Q^η .

Formulation of the problem. We consider Maccheroni type objective function

$$J^{\pi,\eta}(x, y, r, t) := \mathbb{E}_{x,y,r,t}^\eta [-X_T^\pi] - y \mathbb{E} \left[\frac{dQ^\eta}{dP} \right]^2 = \mathbb{E}_{x,y,r,t}^\eta [-X_T^\pi - Y_T^\eta].$$

The investors aim is to

$$(2.3) \quad \text{minimize} \quad \sup_{\eta \in \mathcal{M}} J^{\pi,\eta}(x, y, r, t)$$

over a class of admissible strategies $\mathcal{A}_{x,y,r,t}$.

As usually, the problem (2.3) might be considered as a zero-sum stochastic differential game problem. We are looking for a saddle point $(\pi^*, \eta^*) \in \mathcal{A}_{x,y,r,t} \times \mathcal{M}$ and a value function $V(x, y, r, t)$ such that

$$J^{\pi^*, \eta}(x, y, r, t) \leq J^{\pi^*, \eta^*}(x, y, r, t) \leq J^{\pi, \eta^*}(x, y, r, t)$$

and

$$V(x, y, r, t) = J^{\pi^*, \eta^*}(x, y, r, t).$$

The verification theorem. As announced we solve the problem by applying stochastic control theory. Let us remind, that

$$(2.4) \quad \begin{cases} dX_t^\pi = (\pi_t \lambda(r_t) \sigma(r_t, t) + r_t X_t^\pi) dt + \pi_t \sigma(r_t, t) dW_t, \\ dY_t^\eta = \eta_t Y_t^\eta dW_t, \\ dr_t = \bar{\mu}(r_t) dt + \bar{\sigma}(r_t) dW_t. \end{cases}$$

It is convenient to consider Q^η -dynamics of system (2.4). After applying the Girsanov transformation, we have

$$\begin{cases} dX_t^\pi = (\pi_t \sigma(r_t, t) (\lambda(r_t) + \eta_t) + r_t X_t^\pi) dt + \pi_t \sigma(r_t, t) dW_t^\eta, \\ dY_t^\eta = \eta_t^2 Y_t^\eta dt + \eta_t Y_t^\eta dW_t^\eta, \\ dr_t = (\bar{\mu}(r_t) + \bar{\sigma}(r_t) \eta_t) dt + \bar{\sigma}(r_t) dW_t^\eta, \end{cases}$$

where $(W_t^\eta, 0 \leq t \leq T)$ is Q^η -Brownian motion defined as

$$dW_t^\eta = dW_t - \eta_t dt.$$

Let $\mathcal{L}^{\pi, \eta}$ be the differential operator given by

$$\begin{aligned} \mathcal{L}^{\pi, \eta} V(x, y, r, t) := & V_t + (\pi \sigma(r, t) (\lambda(r) + \eta) + r x) V_x + \eta^2 y V_y + (\bar{\mu}(r) + \bar{\sigma}(r) \eta) V_r \\ & + \frac{1}{2} \pi^2 \sigma^2(r, t) V_{xx} + \frac{1}{2} \eta^2 y^2 V_{yy} + \frac{1}{2} \bar{\sigma}^2(r) V_{rr} \\ & + \pi \sigma(r, t) \eta y V_{xy} + \pi \sigma(r, t) \bar{\sigma}(r) V_{xr} + \eta \bar{\sigma}(r) y V_{yr}. \end{aligned}$$

We can now formulate the Verification Theorem. The proof of this theorem is exactly the same as the proof of analogous theorem from Trybuła and Zawisza [17], so in this paper we omit it.

Theorem 2.2 (Verification Theorem). *Suppose there exists a function*

$$V \in \mathcal{C}^{2,2,2,1}(\mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T]) \cap \mathcal{C}(\mathbb{R} \times [0, +\infty) \times \mathbb{R} \times [0, T])$$

and a Markov control

$$(\pi^*, \eta^*) \in \mathcal{A}_{x,y,r,t} \times \mathcal{M},$$

such that

$$(2.5) \quad \mathcal{L}^{\pi^*(x,y,r,t), \eta^*} V(x, y, r, t) \leq 0,$$

$$(2.6) \quad \mathcal{L}^{\pi, \eta^*(x,y,r,t)} V(x, y, r, t) \geq 0,$$

$$(2.7) \quad \mathcal{L}^{\pi^*(x,y,r,t), \eta^*(x,y,r,t)} V(x, y, r, t) = 0,$$

$$(2.8) \quad V(x, y, r, T) = -x - y$$

$$\text{for all } \eta \in \mathbb{R}, \pi \in \mathbb{R}, (x, y, r, t) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T],$$

and

$$(2.9) \quad \mathbb{E}_{x,y,r,t}^\eta \left[\sup_{t \leq s \leq T} |V(X_s^\pi, Y_s^\eta, r_s, s)| \right] < +\infty$$

$$\text{for all } (x, y, r, t) \in \mathbb{R} \times [0, +\infty) \times \mathbb{R} \times [0, T], \pi \in \mathcal{A}_{x,y,r,t}, \eta \in \mathcal{M}.$$

Then

$$J^{\pi^*, \eta}(x, y, r, t) \leq V(x, y, r, t) \leq J^{\pi, \eta^*}(x, y, r, t) \\ \text{for all } \pi \in \mathcal{A}_{x, y, r, t}, \eta \in \mathcal{M},$$

and

$$V(x, y, r, t) = J^{\pi^*, \eta^*}(x, y, r, t).$$

Solution to the minimax problem. To find the saddle point we start with analyzing a Hamilton-Jacobi-Bellman-Isaacs equation

$$(2.10) \quad \min_{\pi \in \mathbb{R}} \max_{\eta \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, r, t) = 0,$$

i.e.

$$V_t + rxV_x + \bar{\mu}(r)V_r + \frac{1}{2}\bar{\sigma}^2(r)V_{rr} \\ + \min_{\pi \in \mathbb{R}} \max_{\eta \in \mathbb{R}} \left\{ \pi\sigma(r, t)(\lambda(r) + \eta)V_x + \eta^2yV_y + \bar{\sigma}(r)\eta V_r + \frac{1}{2}\pi^2\sigma^2(r)V_{xx} \right. \\ \left. + \frac{1}{2}\eta^2y^2V_{yy} + \pi\sigma(r, t)\eta yV_{xy} + \pi\sigma(r, t)\bar{\sigma}(r)V_{xr} + \eta\bar{\sigma}(r)yV_{yr} \right\} = 0.$$

We expect $V(x, y, r, t)$ to be of the form

$$(2.11) \quad V(x, y, r, t) = H(r, t)x + G(r, t)y,$$

where

$$H(r, T) = -1 \quad \text{and} \quad G(r, T) = -1.$$

Then we have

$$xH_t + yG_t + rxH + \bar{\mu}(r)(xH_r + yG_r) + \frac{1}{2}\bar{\sigma}^2(r)(xH_{rr} + yG_{rr}) \\ + \min_{\pi \in \mathbb{R}} \max_{\eta \in \mathbb{R}} \left\{ \pi\sigma(r, t)(\lambda(r) + \eta)H + \eta^2yG \right. \\ \left. + \bar{\sigma}(r)(\eta x + \pi\sigma(r, t))H_r + 2\eta\bar{\sigma}(r)yG_r \right\} = 0.$$

The maximum over η is attained at $\eta^*(\pi)$, where

$$\eta^*(\pi) = -\frac{\sigma(r, t)H}{2yG}\pi - \frac{\bar{\sigma}(r)(xH_r + 2yG_r)}{2yG}.$$

For $\eta^*(\pi)$ our equation is of the form

$$(2.12) \quad xH_t + yG_t + rxH + \bar{\mu}(r)(xH_r + yG_r) + \frac{1}{2}\bar{\sigma}^2(r)(xH_{rr} + yG_{rr}) \\ + \min_{\pi \in \mathbb{R}} \left\{ \pi\sigma(r, t)(\lambda(r) + \eta^*(\pi))H + (\eta^*(\pi))^2yG \right. \\ \left. + \bar{\sigma}(r)(\eta^*(\pi)x + \pi\sigma(r, t))H_r + 2\eta^*(\pi)\bar{\sigma}(r)yG_r \right\} = 0.$$

The minimum over π is attained at

$$(2.13) \quad \pi^* = 2yG \left[\frac{\lambda(r)}{\sigma(r,t)} \frac{1}{H} + \frac{\bar{\sigma}(r)}{\sigma(r,t)} \left(\frac{H_r}{H^2} - \frac{G_r}{GH} \right) \right] - x \frac{\bar{\sigma}(r)}{\sigma(r,t)} \frac{H_r}{H}.$$

It is worth to notice here that

$$(2.14) \quad \eta^*(\pi^*) = -\lambda(r) - \bar{\sigma}(r) \frac{H_r}{H},$$

so the saddle point candidate

$$(2.15) \quad (\pi^*, \eta^*(\pi^*))$$

looks as follows

$$\begin{aligned} \pi^* &= 2yG \left[\frac{\lambda(r)}{\sigma(r,t)} \frac{1}{H} + \frac{\bar{\sigma}(r)}{\sigma(r,t)} \left(\frac{H_r}{H^2} - \frac{G_r}{GH} \right) \right] - x \frac{\bar{\sigma}(r)}{\sigma(r,t)} \frac{H_r}{H}, \\ \eta^*(\pi^*) &= -\lambda(r) - \bar{\sigma}(r) \frac{H_r}{H}. \end{aligned}$$

Now we substitute (2.13) into (2.12) and get the final equation of the form

$$\begin{aligned} x \left[H_t + rH + (\bar{\mu}(r) - \bar{\sigma}(r)\lambda(r)) H_r + \frac{1}{2} \bar{\sigma}^2(r) H_{rr} - \bar{\sigma}^2(r) \frac{H_r^2}{H} \right] \\ + y \left[G_t + \frac{1}{2} \bar{\sigma}^2(r) G_{rr} + \left(\lambda(r) + \bar{\sigma}(r) \frac{H_r}{H} \right)^2 G \right. \\ \left. + \left(\bar{\mu}(r) - 2\bar{\sigma}(r)\lambda(r) - 2\bar{\sigma}^2(r) \frac{H_r}{H} \right) G_r \right] = 0. \end{aligned}$$

Therefore, instead of solving completely nonlinear equations it is sufficient to find a classical (class $\mathcal{C}^{2,1}$) unique solutions for two semilinear equations:

$$(2.16) \quad H_t + rH + (\bar{\mu}(r) - \bar{\sigma}(r)\lambda(r)) H_r + \frac{1}{2} \bar{\sigma}^2(r) H_{rr} - \bar{\sigma}^2(r) \frac{H_r^2}{H} = 0,$$

with terminal condition $H(r, T) = -1$ and

$$(2.17) \quad \begin{aligned} G_t + \frac{1}{2} \bar{\sigma}^2(r) G_{rr} + \left(\lambda(r) + \bar{\sigma}(r) \frac{H_r}{H} \right)^2 G \\ + \left(\bar{\mu}(r) - 2\bar{\sigma}(r)\lambda(r) - 2\bar{\sigma}^2(r) \frac{H_r}{H} \right) G_r = 0, \end{aligned}$$

with terminal condition $G(r, T) = -1$. We will back to these two equations in Section 3.

Now we need to prove two usefull lemmas.

Lemma 2.3. *Suppose that function $V \in \mathcal{C}^{2,2,1}(\mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T))$ given by (2.11) is a classical unique solution to (2.10). Moreover, let $(\pi^*, \eta^*(\pi^*)) \in \mathcal{A}_{x,y,r,t} \times \mathcal{M}$ be determined using (2.15). Then conditions (2.5) - (2.8) of Theorem 2.2 are satisfied.*

Proof. We already know that

$$\max_{\eta \in \mathbb{R}} \mathcal{L}^{\pi^*, \eta} V(x, y, r, t) = 0, \quad \mathcal{L}^{\pi^*, \eta^*} V(x, y, r, t) = 0$$

and

$$V(x, y, r, T) = -x - y,$$

which confirms (2.5), (2.7) and (2.8).

To prove (2.6) it is sufficient to use (2.12) and (2.14) and simply verify that

$$\min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta^*(\pi^*)} V(x, y, r, t) = 0.$$

□

The second lemma will be helpful in Section 4 to prove the main result.

Lemma 2.4. *Suppose that initial conditions (x_0, y_0, r_0, t_0) are fixed, the saddle point*

$$(\pi^*, \eta^*(\pi^*)) \in \mathcal{A}_{x_0, y_0, r_0, t_0} \times \mathcal{M}$$

is given by (2.15), $H(r, t)$ and $G(r, t)$ are a classical unique solutions to equations (2.16) and (2.17) respectively. Then

$$2G(r_t, t)Y_t^{\eta^*} = 2G(r_0, t_0)y_0 + H(r_0, t_0)x_0 - H(r_t, t)X_t^{\pi^*}, \quad \forall t \in [t_0, T].$$

Proof. It is sufficient to prove only that

$$d(H(r_t, t)X_t^{\pi^*}) = d(-2G(r_t, t)Y_t^{\eta^*}).$$

First of all, note that for saddle point given by (2.15) system of equations (2.4) is of the form

$$\begin{aligned} dX_t^{\pi^*} = & \left\{ 2Y_t^{\eta^*} G \left[\frac{\lambda^2(r_t)}{H} + \bar{\sigma}(r_t)\lambda(r_t) \left(\frac{H_r}{H^2} - \frac{G_r}{GH} \right) \right] - X_t^{\pi^*} \lambda(r_t) \bar{\sigma}(r_t) \frac{H_r}{H} + r_t X_t^{\pi^*} \right\} dt \\ & + \left\{ 2Y_t^{\eta^*} G \left[\frac{\lambda(r_t)}{H} + \bar{\sigma}(r_t) \left(\frac{H_r}{H^2} - \frac{G_r}{GH} \right) \right] - X_t^{\pi^*} \bar{\sigma}(r_t) \frac{H_r}{H} \right\} dW_t \end{aligned}$$

and

$$dY_t^{\eta^*} = - \left\{ \lambda(r_t) + \bar{\sigma}(r_t) \frac{H_r}{H} \right\} Y_t^{\eta^*} dW_t.$$

Using (2.16) we can verify that

$$dH(r_t, t) = \left[-r_t H(r_t, t) + \bar{\sigma}(r_t) \lambda(r_t) H_r(r_t, t) + \bar{\sigma}^2(r_t) \frac{H_r^2(r_t, t)}{H(r_t, t)} \right] dt + \bar{\sigma}(r_t) H_r(r_t, t) dW_t.$$

Moreover, we have

$$d(H(r_t, t)X_t^{\pi^*}) = H(r_t, t)dX_t^{\pi^*} + X_t^{\pi^*}dH(r_t, t) + dH(r_t, t)dX_t^{\pi^*},$$

so substituting the appropriate dynamics to above equation we get

$$(2.18) \quad d(H(r_t, t)X_t^{\pi^*}) = 2Y_t^{\eta^*} \left\{ \left[\left(\lambda(r_t) + \bar{\sigma}(r_t) \frac{H_r(r_t, t)}{H(r_t, t)} \right)^2 G(r_t, t) - \bar{\sigma}(r_t) \left(\lambda(r_t) + \bar{\sigma}(r_t) \frac{H_r(r_t, t)}{H(r_t, t)} \right) G_r(r_t, t) \right] dt + \left[\left(\lambda(r_t) + \bar{\sigma}(r_t) \frac{H_r(r_t, t)}{H(r_t, t)} \right) G(r_t, t) - \bar{\sigma}(r_t) G_r(r_t, t) \right] dW_t \right\}.$$

Now using (2.17) we can verify that

$$dG(r_t, t) = \left[2\bar{\sigma}(r_t) \left(\lambda(r_t) + \bar{\sigma}(r_t) \frac{H_r(r_t, t)}{H(r_t, t)} \right) G_r(r_t, t) - \left(\lambda(r_t) + \bar{\sigma}(r_t) \frac{H_r(r_t, t)}{H(r_t, t)} \right)^2 G(r_t, t) \right] dt + \bar{\sigma}(r_t) G_r(r_t, t) dW_t.$$

Moreover, we have

$$d \left(-2G(r_t, t)Y_t^{\eta^*} \right) = -2G(r_t, t)dY_t^{\eta^*} - 2Y_t^{\eta^*}dG(r_t, t) - 2dG(r_t, t)dY_t^{\eta^*},$$

so substituting the appropriate dynamics to above equation we get the right hand side of (2.18). \square

Remark 2.5. Note that process $(Y_t^{\eta^*}, t_0 \leq t \leq T)$ is not directly observable but fortunately the above lemma ensures that for fixed initial conditions (x_0, y_0, r_0, t_0) instead of Markov strategy

$$\pi^* = 2Y_t^{\eta^*} G \left[\frac{\lambda(r_t)}{\sigma(r_t, t)} \frac{1}{H} + \frac{\bar{\sigma}(r_t)}{\sigma(r_t, t)} \left(\frac{H_r}{H^2} - \frac{G_r}{GH} \right) \right] - X_t^{\pi^*} \frac{\bar{\sigma}(r_t)}{\sigma(r_t, t)} \frac{H_r}{H},$$

we can use

$$\begin{aligned} \hat{\pi}^* &= [2G(r_0, t_0)y_0 + H(r_0, t_0)x_0 - H(r_t, t)X_t^{\pi^*}] \\ &\quad * \left[\frac{\lambda(r_t)}{\sigma(r_t, t)} \frac{1}{H} + \frac{\bar{\sigma}(r_t)}{\sigma(r_t, t)} \left(\frac{H_r}{H^2} - \frac{G_r}{GH} \right) \right] - X_t^{\pi^*} \frac{\bar{\sigma}(r_t)}{\sigma(r_t, t)} \frac{H_r}{H}. \end{aligned}$$

3. CLASSICAL SMOOTH SOLUTIONS TO RESULTING EQUATIONS

In this section we give a set of assumptions to ensure existence of a classical (class $\mathcal{C}^{2,1}$) unique solutions to equations (2.16) and (2.17) with appropriate terminal conditions.

To solve equation (2.16) with boundary condition $H(r, T) = -1$ the following ansatz is made

$$H(r, t) = -\Gamma^a(r, t), \quad \text{where} \quad \Gamma(r, T) = 1 \quad \text{and} \quad \Gamma(r, t) > 0,$$

to obtain

$$\Gamma_t + \frac{r}{a}\Gamma + (\bar{\mu}(r) - \bar{\sigma}(r)\lambda(r))\Gamma_r + \frac{1}{2}\bar{\sigma}^2(r)\Gamma_{rr} + \left[\frac{1}{2}(a-1) - a \right] \bar{\sigma}^2(r) \frac{\Gamma_r^2}{\Gamma} = 0.$$

Note that for $a = -1$ we have

$$(3.1) \quad \Gamma_t - r\Gamma + (\bar{\mu}(r) - \bar{\sigma}(r)\lambda(r))\Gamma_r + \frac{1}{2}\bar{\sigma}^2(r)\Gamma_{rr} = 0.$$

Nevertheless, it is convenient to make another substitution. Namely, if we substitute

$$\Gamma(r, t) = e^{-r(T-t)}F(r, t), \quad \text{where} \quad F(r, T) = 1 \quad \text{and} \quad F(r, t) > 0,$$

we get the following equation

$$(3.2) \quad F_t + \left[\frac{1}{2}\bar{\sigma}^2(r)(T-t)^2 - (T-t)(\bar{\mu}(r) - \bar{\sigma}(r)\lambda(r)) \right] F \\ + [\bar{\mu}(r) - \bar{\sigma}(r)\lambda(r) - \bar{\sigma}^2(r)(T-t)] F_r + \frac{1}{2}\bar{\sigma}^2(r)F_{rr} = 0.$$

Now we are going to prove existence of a classical (class $\mathcal{C}^{2,1}$) unique solution to equation (3.2) with boundary condition $F(r, T) = 1$.

Remark 3.1. From Theorem 4.6, Chapter 6 of Friedman [7], it follows that if $\bar{\mu}$, $\bar{\sigma}$ and $\bar{\sigma} \cdot \lambda$ are Lipschitz continuous and bounded, $\bar{\sigma}^2 > \varepsilon > 0$, then there exists a classical (class $\mathcal{C}^{2,1}(\mathbb{R} \times [0, T]) \cap \mathcal{C}(\mathbb{R} \times [0, T])$) unique solution F to equation (3.2) which is bounded (see in addition estimate 4.12, Chapter 6 of Friedman [7]). It is well known that such solution satisfies the Feynman-Kac representation:

$$F(r, t) = \mathbb{E}_{r,t}^{\tilde{P}} \left[\exp \left\{ \int_t^T \phi(\tilde{r}_s, s) ds \right\} \right],$$

where

$$\phi(r, t) := \frac{1}{2}\bar{\sigma}^2(r)(T-t)^2 - (T-t)(\bar{\mu}(r) - \bar{\sigma}(r)\lambda(r)),$$

$(\tilde{W}_s, t \leq s \leq T)$ is a Brownian motion with respect to \tilde{P} and

$$d\tilde{r}_s = [\bar{\mu}(\tilde{r}_s) - \bar{\sigma}(\tilde{r}_s)\lambda(\tilde{r}_s) - \bar{\sigma}^2(\tilde{r}_s)(T-t)] ds + \bar{\sigma}(\tilde{r}_s)d\tilde{W}_s, \quad \tilde{r}_t = r.$$

Moreover, note that ϕ is bounded and Lipschitz continuous function. It means that F is bounded and bounded away from zero.

To prove other properties of F we need two more lemmas.

Lemma 3.2. Suppose that process $(h_t, 0 \leq t \leq T)$ with deterministic starting point is given by

$$dh(t) = \zeta_1(t)dt + \zeta_2(t)d\tilde{W}_t,$$

where ζ_1 and ζ_2 are bounded stochastic processes and $(\tilde{W}_t, 0 \leq t \leq T)$ is a Brownian motion with respect to \tilde{P} . Then

$$\mathbb{E}^{\tilde{P}} \left[\sup_{0 \leq t \leq T} \exp \left\{ \int_0^t h(s) ds \right\} \right] < +\infty.$$

Proof. Note that

$$d(th(t)) = (h(t) + t\zeta_1(t))dt + t\zeta_2(t)d\tilde{W}_t,$$

which can be rewritten into

$$th(t) = \int_0^t h(s)ds + \int_0^t s\zeta_1(s)ds + \int_0^t s\zeta_2(s)d\tilde{W}_s.$$

It means that

$$\mathbb{E}^{\tilde{P}} \left[\sup_{0 \leq t \leq T} \exp \left\{ \int_0^t h(s)ds \right\} \right] = \mathbb{E}^{\tilde{P}} \left[\sup_{0 \leq t \leq T} \exp \left\{ th(t) - \int_0^t s\zeta_1(s)ds - \int_0^t s\zeta_2(s)d\tilde{W}_s \right\} \right].$$

Moreover, we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} \exp \left\{ th(t) - \int_0^t s\zeta_1(s)ds - \int_0^t s\zeta_2(s)d\tilde{W}_s \right\} \\ & \leq \sup_{0 \leq t \leq T} \mathbb{1}_{\{h(t) < 0\}} \exp \left\{ th(t) - \int_0^t s\zeta_1(s)ds - \int_0^t s\zeta_2(s)d\tilde{W}_s \right\} \\ & \quad + \sup_{0 \leq t \leq T} \mathbb{1}_{\{h(t) \geq 0\}} \exp \left\{ th(t) - \int_0^t s\zeta_1(s)ds - \int_0^t s\zeta_2(s)d\tilde{W}_s \right\} \\ & \leq \sup_{0 \leq t \leq T} \exp \left\{ - \int_0^t s\zeta_1(s)ds - \int_0^t s\zeta_2(s)d\tilde{W}_s \right\} \\ & \quad + \sup_{0 \leq t \leq T} \exp \left\{ Th(t) - \int_0^t s\zeta_1(s)ds - \int_0^t s\zeta_2(s)d\tilde{W}_s \right\}. \end{aligned}$$

This concludes the proof since both processes under supremum are solutions to linear equations with bounded coefficients. \square

Lemma 3.3. *Suppose $\bar{\mu}$, $\bar{\sigma}$ and $\bar{\sigma} \cdot \lambda$ are Lipschitz continuous and bounded, $\bar{\sigma}^2 > \varepsilon > 0$ and F is a bounded solution to equation (3.2). Then the first and the second r -derivative of F are bounded.*

Proof. To get a bound for F_r it is sufficient to estimate the Lipschitz constant. First of all, note that for $r_1, r_2 \in (-\infty, a]$ there exists $L_a > 0$ such that

$$(3.3) \quad |e^{r_1} - e^{r_2}| \leq L_a |r_1 - r_2|.$$

Secondly using (3.3) and the fact that from Remark 3.1 function $\phi(r, t)$ is bounded and Lipschitz continuous we obtain existence of $L > 0$ that

$$\begin{aligned} |F(r, t) - F(\bar{r}, t)| & \leq L \mathbb{E}^{\tilde{P}} \left[\int_t^T |\tilde{r}_s(r, t) - \tilde{r}_s(\bar{r}, t)| ds \right] \\ & \leq LT \mathbb{E}^{\tilde{P}} \left[\sup_{t \leq s \leq T} |\tilde{r}_s(r, t) - \tilde{r}_s(\bar{r}, t)| \right], \end{aligned}$$

where from notational convenience we wrote $\mathbb{E}^{\tilde{P}} f(\tilde{r}_s(r, t))$ instead of $\mathbb{E}_{r,t}^{\tilde{P}} f(\tilde{r}_s)$. Now, it is well known (Theorem 1.3.16 from Pham [16]) there exists $C_T > 0$ such that

$$\mathbb{E}^{\tilde{P}} \left[\sup_{t \leq s \leq T} |\tilde{r}_s(r, t) - \tilde{r}_s(\bar{r}, t)| \right] \leq C_T |r - \bar{r}|.$$

To prove boundness of F_{rr} we first estimate F_t and use the fact that F is a solution to equation (3.2). Let us recall that

$$\Gamma(r, t) = e^{-r(T-t)} F(r, t)$$

is a solution to (3.1). Suppose that $t \leq T$ is fixed and define function

$$v(r, k) = \Gamma(r, k + T - t), \quad k \in [0, t].$$

It is straightforward that v is a bounded solution to (3.1) but with terminal condition $v(r, t) = 1$. Lemma 3.2 ensures that Feynman-Kac representation is possible i.e.

$$v(r, 0) = \mathbb{E}_{r,0}^{\tilde{P}} \left[\exp \left\{ - \int_0^t \tilde{r}_s ds \right\} \right] = e^{-rt} \mathbb{E}_{r,0}^{\tilde{P}} \left[\exp \left\{ - \int_0^t g(s) ds \right\} \right],$$

where $(\tilde{W}_s, 0 \leq s \leq t)$ is a Brownian motion with respect to \tilde{P} ,

$$d\tilde{r}_s = (\bar{\mu}(\tilde{r}_s) - \bar{\sigma}(\tilde{r}_s)\lambda(\tilde{r}_s)) dt + \bar{\sigma}(\tilde{r}_s) d\tilde{W}_s, \quad \tilde{r}_0 = r$$

and

$$g(t) = \int_0^t (\bar{\mu}(\tilde{r}_s) - \bar{\sigma}(\tilde{r}_s)\lambda(\tilde{r}_s)) ds + \int_0^t \bar{\sigma}(\tilde{r}_s) d\tilde{W}_s.$$

Now, note that

$$F(r, T - t) = e^{rt} v(r, 0),$$

so after differentiating F with respect to t we get

$$|F_t(r, T - t)| \leq \mathbb{E}_{r,0}^{\tilde{P}} \left[|g(t)| \exp \left\{ - \int_0^t g(s) ds \right\} \right].$$

Differentiation is possible since

$$\begin{aligned} & \mathbb{E}_{r,0}^{\tilde{P}} \left[\sup_{0 \leq t \leq T} \left(|g(t)| \exp \left\{ - \int_0^t g(s) ds \right\} \right) \right] \\ & \leq \frac{1}{2} \mathbb{E}_{r,0}^{\tilde{P}} \left[\sup_{0 \leq t \leq T} |g(t)|^2 \right] + \frac{1}{2} \mathbb{E}_{r,0}^{\tilde{P}} \left[\sup_{0 \leq t \leq T} \exp \left\{ -2 \int_0^t g(s) ds \right\} \right] \end{aligned}$$

and from Lemma 3.2 we know that

$$\mathbb{E}_{r,0}^{\tilde{P}} \left[\sup_{0 \leq t \leq T} \exp \left\{ -2 \int_0^t g(s) ds \right\} \right] < +\infty.$$

□

Now we are ready to consider the second equation (2.17). Note that

$$(3.4) \quad \frac{H_r}{H} = -\frac{\Gamma_r}{\Gamma} = (T-t) - \frac{F_r}{F}$$

and equation (2.17) has the following form

$$(3.5) \quad G_t + \frac{1}{2}\bar{\sigma}^2(r)G_{rr} + \left(\lambda(r) + \bar{\sigma}(r)\left[(T-t) - \frac{F_r}{F}\right]\right)^2 G \\ + \left(\bar{\mu}(r) - 2\bar{\sigma}(r)\lambda(r) - 2\bar{\sigma}^2(r)\left[(T-t) - \frac{F_r}{F}\right]\right) G_r = 0.$$

Remark 3.4. Under conditions of Lemma 3.3 we get boundness and Lipschitz continuity of (3.4). This ensures that there exists a classical (class $\mathcal{C}^{2,1}(\mathbb{R} \times [0, T]) \cap \mathcal{C}(\mathbb{R} \times [0, T])$) unique solution to (3.5), which satisfy Feynman-Kac representation:

$$G(r, t) = \mathbb{E}_{r,t}^{\hat{P}} \left[\exp \left\{ \int_t^T \psi(\hat{r}_s, s) ds \right\} \right],$$

where

$$\psi(r, t) = \left(\lambda(r) + \bar{\sigma}(r) \left[(T-t) - \frac{F_r}{F} \right] \right)^2,$$

$(\hat{W}_s, t \leq s \leq T)$ is a Brownian motion with respect to \hat{P} and

$$d\hat{r}_s = \left(\bar{\mu}(\hat{r}_s) - 2\bar{\sigma}(\hat{r}_s)\lambda(\hat{r}_s) - 2\bar{\sigma}^2(\hat{r}_s) \left[(T-s) - \frac{F_r}{F} \right] \right) ds + \bar{\sigma}(\hat{r}_s)d\hat{W}_s, \quad \hat{r}_t = r.$$

At the end it is worth noticing that since ψ is Lipschitz continuous in r (uniformly wrt. t) and bounded, then function G is bounded, bounded away from zero and first r -derivative of G is bounded (see proof of Lemma 3.3).

4. FINAL SOLUTION

Theorem 4.1. Suppose $\bar{\sigma}^2 > \varepsilon > 0$ and $\bar{\mu}$, $\bar{\sigma}$ and $\bar{\sigma} \cdot \lambda$ are Lipschitz continuous and bounded. Then for each initial conditions (x_0, y_0, r_0, t_0) there exists a Markov saddle point

$$(4.1) \quad (\pi^*, \eta^*) \in \mathcal{A}_{x_0, y_0, r_0, t_0} \times \mathcal{M}$$

for problem (2.3) such that

$$\pi^* = 2Y_t^{\eta^*} G \left[\frac{\lambda(r_t)}{\sigma(r_t, t)} \frac{1}{H} + \frac{\bar{\sigma}(r_t)}{\sigma(r_t, t)} \left(\frac{H_r}{H^2} - \frac{G_r}{GH} \right) \right] - X_t^{\pi^*} \frac{\bar{\sigma}(r_t)}{\sigma(r_t, t)} \frac{H_r}{H}, \\ \eta^* = -\lambda(r_t) - \bar{\sigma}(r_t) \frac{H_r}{H},$$

where G and H are a classical unique solutions to (2.16) and (2.17) respectively.

Proof. It follows from Remark 3.1, Lemma 3.3 and Remark 3.4 that there exist a classical unique solutions to (2.16) and (2.17), which are bounded, bounded away from zero and have first r -derivatives bounded.

If we set

$$V(x, y, r, t) := H(r, t)x + G(r, t)y,$$

then it is sufficient to check whether function V and Markov saddle point (4.1) satisfy all conditions of the Verification Theorem. Due to calculations (2.10) - (2.17) and Lemma 2.3, conditions (2.5) - (2.8) are fulfilled. Now, we only have to prove that (π^*, η^*) belongs to the set $\mathcal{A}_{x_0, y_0, r_0, t_0} \times \mathcal{M}$ and condition (2.9) holds, so we have to show that for any $\eta \in \mathcal{M}$

$$(4.2) \quad \mathbb{E}_{x_0, y_0, r_0, t_0}^\eta \left[\sup_{t_0 \leq s \leq T} |V(X_s^{\pi^*}, Y_s^{\eta^*}, r_s, s)| \right] < +\infty$$

and

$$(4.3) \quad \mathbb{E}_{x_0, y_0, r_0, t_0}^\eta \left[\sup_{t_0 \leq s \leq T} |X_s^{\pi^*}| \right] < +\infty.$$

First of all, let us remind that $Y_t^{\eta^*}$ is a solution to equation

$$(4.4) \quad dY_t^{\eta^*} = - \left\{ \lambda(r_t) + \bar{\sigma}(r_t) \frac{H_r}{H} \right\} Y_t^{\eta^*} dW_t.$$

Since G is bounded and (4.4) is a linear stochastic differential equation with bounded coefficients, we have

$$(4.5) \quad \mathbb{E}_{x_0, y_0, r_0, t_0}^\eta \left[\sup_{t_0 \leq s \leq T} |G(r_s, s) Y_s^{\eta^*}| \right] \\ \leq \sqrt{\mathbb{E}^P \left(\frac{dQ^\eta}{dP} \right)^2} \sqrt{\mathbb{E}_{x_0, y_0, r_0, t_0}^P \sup_{t_0 \leq s \leq T} |G(r_s, s) Y_s^{\eta^*}|^2} < +\infty \quad \forall \eta \in \mathcal{M}.$$

To prove the same with $H(r_t, t) X_t^{\pi^*}$ we use Lemma 2.4. Namely, let us remind that for fixed initial conditions (x_0, y_0, r_0, t_0) we have

$$(4.6) \quad H(r_t, t) X_t^{\pi^*} = 2G(r_0, t_0) y_0 + H(r_0, t_0) x_0 - 2G(r_t, t) Y_t^{\eta^*}.$$

Since G and $Y_t^{\eta^*}$ satisfy (4.5), then

$$\mathbb{E}_{x_0, y_0, r_0, t_0}^\eta \left[\sup_{t_0 \leq s \leq T} |H(r_s, s) X_s^{\eta^*}| \right] < +\infty \quad \forall \eta \in \mathcal{M}$$

and condition (4.2) holds.

To prove (4.3) first we show that for any deterministic continuous function $w(t)$ and for any $\eta \in \mathcal{M}$ we have

$$\mathbb{E}_{r_0, t_0}^\eta \left[\sup_{t_0 \leq t \leq T} e^{w(t)r_t} \right] < +\infty,$$

where $(r_t, t_0 \leq t \leq T)$ is a stochastic process given by

$$dr_t = \bar{\mu}(r_t)dt + \bar{\sigma}(r_t)dW_t.$$

It is easy to see, that

$$r_t = r_0 + \int_{t_0}^t \bar{\mu}(r_s) ds + \int_{t_0}^t \bar{\sigma}(r_s) dW_s$$

and consequently, since e^{r_t} is a solution to linear stochastic equation with bounded coefficients, we have

$$(4.7) \quad \mathbb{E}_{r_0, t_0}^P \left[\sup_{t_0 \leq t \leq T} e^{r_t} \right] < +\infty.$$

Now, observe that

$$\begin{aligned} \sup_{t_0 \leq t \leq T} e^{w(t)r_t} &\leq \sup_{t_0 \leq t \leq T} \mathbb{1}_{\{r_t < 0\}} e^{w(t)r_t} + \sup_{t_0 \leq t \leq T} \mathbb{1}_{\{r_t \geq 0\}} e^{w(t)r_t} \\ &\leq \sup_{t_0 \leq t \leq T} \mathbb{1}_{\{r_t < 0\}} e^{\underline{w}r_t} + \sup_{t_0 \leq t \leq T} \mathbb{1}_{\{r_t \geq 0\}} e^{\bar{w}r_t} \\ &\leq \sup_{t_0 \leq t \leq T} e^{\underline{w}r_t} + \sup_{t_0 \leq t \leq T} e^{\bar{w}r_t}, \end{aligned}$$

where

$$\underline{w} = \min_{t_0 \leq t \leq T} w(t) \quad \text{and} \quad \bar{w} = \max_{t_0 \leq t \leq T} w(t).$$

Thus, using Hölder inequality and (4.7), we obtain

$$\mathbb{E}_{r_0, t_0}^\eta \left[\sup_{t_0 \leq t \leq T} e^{w(t)r_t} \right] \leq \sqrt{\mathbb{E}^P \left(\frac{dQ^\eta}{dP} \right)^2} \sqrt{\mathbb{E}_{r_0, t_0}^P \left[\sup_{t_0 \leq t \leq T} e^{2w(t)r_t} \right]} < +\infty \quad \forall \eta \in \mathcal{M}.$$

Finally, if we divide equation (4.6) by $H(r_t, t)$ and remembering that

$$\frac{1}{H(r, t)} = -e^{-(T-t)r} F(r, t),$$

F is bounded, G and $Y_t^{\eta*}$ satisfy (4.5) and

$$\mathbb{E}_{r_0, t_0}^\eta \left[\sup_{t_0 \leq t \leq T} e^{-(T-t)r_t} \right] < +\infty \quad \forall \eta \in \mathcal{M},$$

we get that (4.3) holds. □

5. EXAMPLE - VASICEK MODEL

In this section we solve problem (2.3) in the Vasicek model. Namely, we assume that $\sigma(t) > 0$ is a continuous function, $\alpha > 0$, $\lambda, \bar{\theta}, \bar{\sigma} \in \mathbb{R}$ and

$$\lambda(r) = \lambda, \quad \sigma(r, t) = \sigma(t), \quad \bar{\mu}(r) = \bar{\theta} - \alpha r, \quad \bar{\sigma}(r) = \bar{\sigma}.$$

Since $\bar{\mu}$ is not bounded, the solution is not a straightforward consequence of Theorem 4.1 and needs a separate proof.

Theorem 5.1. *Suppose that initial conditions (x_0, y_0, r_0, t_0) are fixed. In the Vasicek model the saddle point*

$$(5.1) \quad (\pi^*, \eta^*(\pi^*)) \in \mathcal{A}_{x_0, y_0, r_0, t_0} \times \mathcal{M}$$

looks as follows

$$\begin{aligned} \pi^* &= \frac{2Y_t^{\eta^*}}{\sigma(t)} (\lambda + \bar{\sigma} B_1(t)) \frac{A_2(t)}{A_1(t)} e^{-B_1(t)r_t} - X_t^{\pi^*} \frac{\bar{\sigma}}{\sigma(t)} B_1(t), \\ \eta^*(\pi^*) &= -\lambda - \bar{\sigma} B_1(t), \end{aligned}$$

where

$$(5.2) \quad B_1(t) = \frac{1}{\alpha} (1 - e^{-\alpha(T-t)}),$$

$$(5.3) \quad A_1(t) = -\exp \left\{ \int_t^T (\bar{\theta} - \bar{\sigma}\lambda) B_1(s) - \frac{1}{2} \bar{\sigma}^2 B_1^2(s) ds \right\},$$

$$(5.4) \quad A_2(t) = -\exp \left\{ \int_t^T (\lambda + \bar{\sigma} B_1(s))^2 ds \right\}.$$

Proof. Let

$$(5.5) \quad H(r, t) = A_1(t) e^{B_1(t)r}, \quad \text{where} \quad A_1(T) = -1, \quad B_1(T) = 0.$$

Then equation (2.16) is of the form

$$A_1'(t) + r A_1(t) [B_1'(t) - \alpha B_1(t) + 1] + (\bar{\theta} - \bar{\sigma}\lambda) A_1(t) B_1(t) - \frac{1}{2} \bar{\sigma}^2 A_1(t) B_1^2(t) = 0.$$

If $B_1(t)$ looks as (5.2), then we get

$$A_1'(t) + A_1(t) \left[(\bar{\theta} - \bar{\sigma}\lambda) B_1(t) - \frac{1}{2} \bar{\sigma}^2 B_1^2(t) \right] = 0,$$

so $A_1(t)$ is of the form (5.3). Now using (5.5) we can rewrite equation (2.17) as

$$(5.6) \quad G_t + \frac{1}{2} \bar{\sigma}^2 G_{rr} + (\lambda + \bar{\sigma} B_1(t))^2 G + (\bar{\theta} - \alpha r - 2\bar{\sigma}\lambda - 2\bar{\sigma}^2 B_1(t)) G_r = 0.$$

Let

$$(5.7) \quad G(r, t) = A_2(t), \quad \text{where} \quad A_2(T) = -1.$$

Then equation (5.6) looks as

$$A_2'(t) + A_2(t) (\lambda + \bar{\sigma} B_1(t))^2 = 0,$$

so $A_2(t)$ is of the form (5.4). Finally, saddle point candidate (2.15) for H and G given by (5.5) and (5.7) respectively looks as (5.1).

Now we can set

$$V(x, y, r, t) := H(r, t)x + G(r, t)y,$$

and check whether $(\pi^*, \eta^*(\pi^*))$ belongs to the set $\mathcal{A}_{x_0, y_0, r_0, t_0} \times \mathcal{M}$ and condition (2.9) holds. Taking into account the form of H and G and the fact that

$$H(r_t, t)X_t^{\pi^*} = 2G(r_0, t_0)y_0 + H(r_0, t_0)x_0 - 2G(r_t, t)Y_t^{\eta^*},$$

it is sufficient to prove only that for any deterministic continuous function $w(t)$ and for any $\eta \in \mathcal{M}$ we have

$$\mathbb{E}_{r_0, t_0}^{\eta} \left[\sup_{t_0 \leq t \leq T} e^{w(t)r_t} \right] < +\infty,$$

where $(r_t, t_0 \leq t \leq T)$ is a stochastic process given by

$$dr_t = (\bar{\theta} - \alpha r_t)dt + \bar{\sigma}dW_t.$$

If we define

$$\delta_t := e^{\alpha t} r_t,$$

then

$$d\delta_t = \bar{\theta}e^{\alpha t}dt + \bar{\sigma}e^{\alpha t}dW_t,$$

so

$$\delta_t = e^{\alpha t_0} r_{t_0} + \int_{t_0}^t \bar{\theta}e^{\alpha s}ds + \int_{t_0}^t \bar{\sigma}e^{\alpha s}dW_s$$

and note that

$$(5.8) \quad \mathbb{E}_{r_0, t_0}^P \left[\sup_{t_0 \leq t \leq T} e^{\delta_t} \right] < +\infty.$$

Using the same method as in proof of Theorem 4.1 we obtain

$$(5.9) \quad \sup_{t_0 \leq t \leq T} e^{w(t)\delta_t} \leq \sup_{t_0 \leq t \leq T} e^{\underline{w}\delta_t} + \sup_{t_0 \leq t \leq T} e^{\overline{w}\delta_t},$$

where

$$\underline{w} = \min_{t_0 \leq t \leq T} w(t) \quad \text{and} \quad \overline{w} = \max_{t_0 \leq t \leq T} w(t).$$

Finally, taking into account (5.8) and (5.9), we get

$$\begin{aligned} \mathbb{E}_{r_0, t_0}^{\eta} \left[\sup_{t_0 \leq t \leq T} e^{w(t)r_t} \right] &= \mathbb{E}_{r_0, t_0}^{\eta} \left[\sup_{t_0 \leq t \leq T} \exp \{w(t)e^{-\alpha t}\delta_t\} \right] \\ &\leq \sqrt{\mathbb{E}^P \left(\frac{dQ^{\eta}}{dP} \right)^2} \sqrt{\mathbb{E}_{r_0, t_0}^P \left[\sup_{t_0 \leq t \leq T} \exp \{2w(t)e^{-\alpha t}\delta_t\} \right]} < +\infty \quad \forall \eta \in \mathcal{M}. \end{aligned}$$

□

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